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Wavelet correlations in hierarchical branching processes

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ABSTRACT

A study of correlations in tractable multiparticle cascade models in terms of wavelets reveals many promising features. The selfsimilar construction of the wavelet basis functions and their multiscale localization properties provide a new approach to the statistical analysis and analytical control of hierarchically organized branching processes. The exact analytical solution of several discrete models shows that the wavelet transformation supresses redundancy in the correlation information. Wavelet correlations can be naturally interpreted as correlations between structures (clumps) living on different scales.

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1 Introduction

Multiparticle production is often dubbed in terms of branching processes. In this context, selfsimilarity and scaling are particularly appealing concepts since they constitute an intriguingly simple and universal organisation principle of the underlying dynamics. They are found in a large variety of complex phenomena.

Simple discrete hierarchical branching models with selfsimilar density fluctuations were first introduced in the description of energy dissipation in fully developed turbulence [1]-[4]. They combine an independent hierarchical evolution of different branches with a random multiplicative nature of density fluctuations. In multiparticle processes such selfsimilar cascade models serve as paradigmatic toy models for intermittency [5]-[8]. Intermittency is now experimentally well established in soft multiparticle interactions and is now being explored in hard (perturbative) multiparton interactions.

Although these branching models describe local density fluctuations, they are conventionally analyzed (globally) in integrated form as is the case for the experimental data: The underlying concepts of scaling and selfsimilar fluctuations are revealed in a multifractal (moment) analysis [9], which globally averages over the differential (local) correlation structure. A fully differential correlation analysis itself would be a more fundamental approach to reveal selfsimilarity and scaling and deviations from these. However, correlation functions of higher than second order are difficult to measure and to visualize. Nevertheless, we prefer to stick to a fully local description of hierarchical branching models and their analysis in terms of differential correlation densities.

One word about the branching models used in the course of this paper: Alltogether they serve as (oversimplified) discrete approximations to realistic QCD parton cascades in the perturbative regime. We do not intend to overstress this relation, but rather prefer to study new correlation techniques demonstrated by relatively simple branching models. – We restrict the discussion to one-dimensional discrete branching processes only. Higher dimensional branching processes can be treated in the same way as the one-dimensional case and do not change the conclusions drawn. Discreteness which is introduced into the branching process by allowing branching only for discrete "time" steps of the cascade, is also not a fundamental shortcoming compared to continuous processes as long as the step width stays small. We are aware that a complete continuous formulation of the branching process needs a functional approach.

In this paper we essentially convey three messages. First, we generalize the univariate (moment) generating function to a multivariate (correlation density) generating function and derive its evolution equation. Message two unveils a secret unnoticed so far in moment (multifractal) analysis, but which is essential for comparison of model predictions with data: For cascade models with global density fluctuations on top of local density fluctuations, a careful distinction has to be made between moments obtained from a (theoretical) forward evolution of the cascade and moments obtained from an (experimental) backward analysis. Last not least, as message three we propose a choice of a clever basis for the representation of correlation structures for hierarchically organized stochastic processes: wavelets. They lead

to a tremendous simplification and directly unveil information of interest, which is otherwise scattered in the many dimensions of phase space.

The call of wavelets [10]-[14] needs some further introductory remarks: The whole wavelet basis is constructed from dilations and translations of one single "mother" wavelet and represents a selfsimilar and orthogonal basis. Thus we would expect that the wavelet transform uniquely simplifies the correlation functions of selfsimilar processes and, in particular, quasi-diagonalizes the covariance matrix. This has been demonstrated in ref. [15] for fractional Brownian motion and in [16] for the p-model [4], which is the simplest tractable random cascade model. Still more can be learned from the wavelet transform.

To estimate correlation patterns, the human brain follows a strategy different from the standard analysis of correlations: it organizes particles accumulated in densely populated regions into (hard to quantify) "clumps" or "clusters" and unpopulated regions into "voids". If one looks closer into a particular "clump" it may (or may not) again be organized into "clusters" and "voids", but now with respect to the higher (smoothed) background density of the bigger "parent-clump", and so on. – Due to its multiscale localization property the wavelet transformation dissects a (random) signal into contributions from different scales and thus localizes small and large scale structures separately. This feature is the reason why wavelets are so successful in signal analysis [10].

Wavelet correlations provide statistical information about further subclustering of certain structures like clusters, filaments or voids. Correlations between small subclusters living inside larger clusters or voids are naturally revealed in the wavelet transformed correlations, as are correlations between voids and filaments. Loosely speaking, we denote wavelet correlations as "clump" correlations.

In this publication we study conventional and wavelet correlations of one-dimensional hierarchically organized branching processes. For those readers only interested in one or the other part, we have treated conventional and wavelet correlations in separated sections.

In section 2 we define a class of discrete hierarchical branching models. We distinguish between the concepts of deterministic and random branching, which describe a zero lifetime cascade or a cascade with finite lifetime respectively. An evolution equation for the multivariate generating function of the correlation densities is presented, which allows one to calculate the correlation densities recursively. This procedure is exemplified for the so-called p-model [4], α-model [3], the p-model with random branching and a QCD-motivated cascade model [17, 18]. In an additional subsection we emphasize the conceptual distinction between an evolution (= theoretical) and a backward (= experimental) moment analysis.

The concept of wavelet correlations is introduced in section 3. For the same models we discuss the simplified and compressed structure of the wavelet correlations of all orders. On the basis of the p-model we stress the relationship of higher order wavelet correlations with clump correlations. This interpretation is further illustrated as we discuss the wavelet multiresolution analysis of configurations belonging to two-dimensional branching processes.

In section 4, which represents the conclusions, we give an outlook on future applications of the wavelet correlations in multiparticle processes and other fields in physics.

2 Hierarchical cascades – conventional correlation formalism

We study a class of discrete hierarchical cascade models with what we call random or deterministic branching and restrict ourselves to the one-dimensional case only. From a general evolution equation for the generating function we calculate recursion relations for conventional correlation densities. As specific examples, the correlation structure of the frequently used p-model [4, 6] and α-model [3, 5, 7] as well as the p-model with random branching and the QCD-motivated cascade model of ref. [17, 18] are analysed in more detail. We exhibit the importance of the conceptual distinction between moments referring to the forward evolution of the cascade and moments obtained from a backward analysis.

2.1 Hierarchical cascading processes with random branching

We illustrate the topological structure of a hierarchical cascading process with random branching in Fig.1. Starting from a single trunk, the tree might branch into a left and a right branch at the first cascade step with probability $\tilde{p}$; with probability $1-\tilde{p}$ the tree might not branch at the first cascade step. At the second cascade step, every single branch that has formed so far may again split into a left and a right branch with probability $\tilde{p}$ or remains unsplit with probability $1-\tilde{p}$. This prescription is repeated at each further cascade step. For lack of a better terminology we associate *random branching* with the case $\tilde{p} < 1$ and term the case $\tilde{p} = 1$ *deterministic branching*.

In addition to the topological structure of the branching process, the spatial structure is specified by the following rules (see Fig. 2): The origin of the cascading tree corresponds without any loss of generality to an interval $[0, 1]$ with uniform (energy) density ϵ normalized to one, $\epsilon_0^{(0)} = 1$. If the interval does not split with probability $1-\tilde{p}$ at the first cascade step, the energy density will remain the same as before. If, on the other hand, the interval does branch into two halves with probability $\tilde{p}$, one part of the original energy $E = 1$, namely $E_L = \frac{q_L}{2}E = \frac{q_L}{2}$, goes to the left subinterval and another part, namely $E_R = \frac{q_R}{2}$, goes to the right interval. The weights q_L and q_R are random variables in the range $0 \leq q_L, q_R \leq 2$ for both random and determinisitic branching and follow a joint probability distribution $p(q_L, q_R)$, which is often called a "splitting function" or "splitting kernel". Depending on the splitting function $p(q_L, q_R)$, energy may or may not be conserved at one branching step, so that $E_L + E_R = E$ needs not be fulfilled. The energy densities of the resulting left and right subintervals are $\epsilon_0^{(1)} = q_L$ and $\epsilon_1^{(1)} = q_R$ respectively. – This prescription is repeated for all subsequent cascade steps j. Whenever a subinterval splits into two halves, the energy density of the left half subinterval is q_L times the previous energy density, $\epsilon_{2k}^{(j_b+1)} = q_L \epsilon_k^{(j_b)}$, and, correspondingly, the energy density of the right half subinterval is $\epsilon_{2k+1}^{(j_b+1)} = q_R \epsilon_k^{(j_b)}$, where q_L and q_R always follow the same splitting function. The upper index of $\epsilon_k^{(j_b)}$ indicates the number of branchings j_b that occurred on the way to a subinterval and should not be confused with the number of cascade steps j, whereas the lower index numbers the subinterval given

at the branching scale j_b. If the subinterval does not split, the energy density remains the same as before. – This energy curdling is shown in Fig. 2 for one possible realization. Once the last cascade step $j = J$ is reached, the resulting energy density distribution is resolved on this finest scale with 2^J subintervals (= bins). The energy densities $\epsilon_k^{(J)}$, k=0,...,$2^J - 1$, of neighbouring bins then may be identical, whenever a nonbranching has occurred during previous stages of the cascade.

In particle physics, cascade models based on continuous random variables ϵ_k as discussed above are discretized by means of the Poisson transform mechanism [5]. After the last cascade step J, the energy density $\epsilon_k^{(J)}$ is replaced by a discrete particle number n_k which is drawn from a Poissonian with a mean proportional to $\epsilon_k^{(J)}/2^J$. The necessary transition from ordinary to factorial correlation densities is well known, has no impact on our results and will not be further discussed in this paper. As has been demonstrated in ref. [16] this also holds for the transition from ordinary to factorial wavelet correlation densities.

2.2 Evolution equation for conventional correlation densities

The cascading process described in the previous subsection is selfsimilar by construction: At each cascade step the same branching prescription is applied as in the previous steps. As shown in later sections this selfsimilar construction does not necessarily imply perfect scale invariance of correlation functions. Nevertheless, a detailed analysis of correlation densities is the only means to obtain local information on the (selfsimilar) branching structure of a cascade process.

The bin correlation densities are defined as the product of energy densities contained in various bins, averaged or sampled over all possible configurations. For example, one-, two- and three-bin correlation densities read

$$\begin{aligned} \rho_{k_1}^{(J)} &= \langle \epsilon_{k_1}^{(J)} \rangle \quad , \\ \rho_{k_1 k_2}^{(J)} &= \langle \epsilon_{k_1}^{(J)} \epsilon_{k_2}^{(J)} \rangle \quad , \\ \rho_{k_1 k_2 k_3}^{(J)} &= \langle \epsilon_{k_1}^{(J)} \epsilon_{k_2}^{(J)} \epsilon_{k_3}^{(J)} \rangle \quad ; \end{aligned} \tag{1}$$

the brackets $\langle \ldots \rangle$ indicate the averaging over all configurations. The indices k_i run from 0 to $2^J - 1$ and represent the number of different bins at the finest resolution scale J. The correlation densities are most easily determined once the corresponding (characterisitic) generating function

$$Z^{(J)}\left[\vec{\lambda}^{(J)}\right] = \left\langle \exp\left(i \sum_{k=0}^{2^J-1} \lambda_k^{(J)} \epsilon_k^{(J)} \right) \right\rangle \tag{2}$$

is known; they follow by taking appropriate derivatives with respect to the $\lambda_k^{(J)}$:

$$\rho_{k_1}^{(J)} = \frac{1}{i} \frac{\partial Z^{(J)}\left[\vec{\lambda}^{(J)}\right]}{\partial \lambda_{k_1}^{(J)}}\bigg|_{\vec{\lambda}^{(J)}=0} \quad ,$$

$$\rho^{(J)}_{k_1 k_2} = \frac{1}{i^2} \left. \frac{\partial^2 Z^{(J)}\left[\vec{\lambda}^{(J)}\right]}{\partial\lambda^{(J)}_{k_1}\partial\lambda^{(J)}_{k_2}} \right|_{\vec{\lambda}^{(J)}=0} \quad , \tag{3}$$

$$\rho^{(J)}_{k_1 k_2 k_3} = \frac{1}{i^3} \left. \frac{\partial^3 Z^{(J)}\left[\vec{\lambda}^{(J)}\right]}{\partial\lambda^{(J)}_{k_1}\partial\lambda^{(J)}_{k_2}\partial\lambda^{(J)}_{k_3}} \right|_{\vec{\lambda}^{(J)}=0} \quad , \quad \ldots \, .$$

In the following we develop a scheme how to determine the generating function at a given evolution scale from the previous evolution scales. This scale-dependent recursion relation then leads to scale-dependent recursion relations for the bin correlation densities.

Suppose we know the generating function $Z^{(j)}\left[\vec{\lambda}^{(j)}\right]$ after j cascade steps. Then we can determine $Z^{(j+1)}$ after $j+1$ cascade steps by a backward evolution: With probability $(1-\tilde{p})$ there has been no branching in this backward evolution step and with probability $\tilde{p}$ a branching into a left and right branch has occurred, where the split energy densities have been weighted with q_L and q_R respectively. This translates into

$$Z^{(j+1)}\left[\vec{\lambda}^{(j+1)}\right] = (1-\tilde{p})\; Z^{(j)}\left[\vec{\lambda}^{(j)}_M\right] + \tilde{p} \int \mathrm{d}q_L \mathrm{d}q_R \; p(q_L, q_R) \; Z^{(j)}\left[q_L \vec{\lambda}^{(j)}_L\right] Z^{(j)}\left[q_R \vec{\lambda}^{(j)}_R\right] \quad , \tag{4}$$

where the rules

$$\begin{aligned}
\vec{\lambda}^{(j+1)} &= \left(\lambda^{(j+1)}_0, \lambda^{(j+1)}_1, \ldots, \lambda^{(j+1)}_{2^{j+1}-1}\right) \quad , \\
\vec{\lambda}^{(j)}_L &= \left(\lambda^{(j+1)}_0, \lambda^{(j+1)}_1, \ldots, \lambda^{(j+1)}_{2^{j}-1}\right) \quad , \\
\vec{\lambda}^{(j)}_R &= \left(\lambda^{(j+1)}_{2^j}, \lambda^{(j+1)}_{2^j+1}, \ldots, \lambda^{(j+1)}_{2^{j+1}-1}\right) \quad , \\
\vec{\lambda}^{(j)}_M &= \left(\lambda^{(j+1)}_0 + \lambda^{(j+1)}_1, \lambda^{(j+1)}_2 + \lambda^{(j+1)}_3, \ldots, \lambda^{(j+1)}_{2^{j+1}-2} + \lambda^{(j+1)}_{2^{j+1}-1}\right)
\end{aligned} \tag{5}$$

specify the spatial splitting structure of the branching process. The splitting function $p(q_L, q_R)$ reflects the probability that random weights q_L and q_R are assigned to the energy densities of the left and right branch respectively and is normalized to one, i.e. $\int p(q_L, q_R)\mathrm{d}q_L\mathrm{d}q_R = 1$. The nonlinear backward evolution equation (4) allows one to express $Z^{(j+1)}$ in terms of $Z^{(j)}$, which in turn can be expressed in terms of $Z^{(j-1)}$, and so on until the roughest scale $j=0$ is reached, where

$$Z^{(0)}\left[\lambda^{(0)}_0\right] = e^{i\lambda^{(0)}_0} \quad . \tag{6}$$

The evolution equation (4) has been derived in detail in [16] for the case of the p-model with deterministic branching.

It is instructive to rewrite Eq.(4) as

$$\frac{\Delta Z^{(j)}\left[\vec{\lambda}\right]}{\Delta j} = \frac{1}{\tau} \int \mathrm{d}q_L \mathrm{d}q_R \; p(q_L, q_R) \left\{ Z^{(j)}\left[q_L \vec{\lambda}_L\right] Z^{(j)}\left[q_R \vec{\lambda}_R\right] - Z^{(j)}\left[\vec{\lambda}_M\right] \right\} \quad , \tag{7}$$

where $\Delta Z^{(j)}\left[\vec{\lambda}\right] = Z^{(j+1)}\left[\vec{\lambda}^{(j+1)}\right] - Z^{(j)}\left[\vec{\lambda}^{(j)}_M\right]$, $\Delta j = 1$ and $1/\tau = \tilde{p}/\Delta j$ is a branching rate. This form is a discrete analogue to the evolution equations widely used to model QCD

parton showers in the perturbative regime [19]. By this analogy we are motivated to study the mathematical structure of the recursive discrete evolution equation (4) in detail.

From Eq. (4) we derive the recursion relations for the bin correlation densities by taking the relevant derivatives (3) with respect to the $\lambda_k^{(j+1)}$ taking into account the transformations (5). For the one-bin correlation density we deduce:

$$\rho_{k_1}^{(j+1)} = (1-\tilde{p})\rho_{m_1}^{(j)} + \tilde{p}\,\overline{q}\,\rho_{m_2}^{(j)} \quad ; \tag{8}$$

the following four cases have to be distinguished:

- $k_1 \in \{0,\ldots,2^j-1\} = \{L\}$ and $k_1 = 2m$ an even number:
  $m_1 = m = k_1/2$, $m_2 = k_1$, $\overline{q} = \overline{q_L} = \int p(q_L,q_R)\,q_L\,\mathrm{d}q_L\mathrm{d}q_R$,
- $k_1 \in \{L\}$ and $k_1 = 2m+1$ an odd number:
  $m_1 = m = (k_1-1)/2$, $m_2 = k_1$, $\overline{q} = \overline{q_L}$,
- $k_1 \in \{2^j,\ldots,2^{j+1}-1\} = \{R\}$ and $k_1 = 2m$ an even number:
  $m_1 = m = k_1/2$, $m_2 = k_1 - 2^j$, $\overline{q} = \overline{q_R} = \int p(q_L,q_R)\,q_R\,\mathrm{d}q_L\mathrm{d}q_R$,
- $k_1 \in \{R\}$ and $k_1 = 2m+1$ an odd number:
  $m_1 = m = (k_1-1)/2$, $m_2 = k_1 - 2^j$, $\overline{q} = \overline{q_R}$.

Because the splitting function $p(q_L,q_R)$ needs not be symmetric in q_L and q_R we distinguish $\overline{q_L}$ and $\overline{q_R}$. For completeness we state that $\rho_0^{(0)} = 1$, which follows directly from Eqs. (3) and (6). – For the two-bin correlation density we have to distinguish two principal cases. If k_1 and k_2 belong to the same branch, i.e. $k_1,k_2 \in \{L\}$ or $k_1,k_2 \in \{R\}$, we get

$$\rho_{k_1k_2}^{(j+1)} = (1-\tilde{p})\rho_{m_1m_2}^{(j)} + \tilde{p}\,\overline{q^2}\,\rho_{m_3m_4}^{(j)} \tag{9}$$

with $\overline{q^2} = \overline{q_L^2} = \int p(q_L,q_R)\,q_L^2\,\mathrm{d}q_L\mathrm{d}q_R$ or $\overline{q^2} = \overline{q_R^2}$ analogously. If k_1 and k_2 belong to different branches, i.e. $k_1 \in \{L\}$, $k_2 \in \{R\}$ or vice versa, we get

$$\rho_{k_1k_2}^{(j+1)} = (1-\tilde{p})\rho_{m_1m_2}^{(j)} + \tilde{p}\,\overline{q_Lq_R}\,\rho_{m_3}^{(j)}\rho_{m_4}^{(j)} \tag{10}$$

with $\overline{q_Lq_R} = \int p(q_L,q_R)\,q_Lq_R\,\mathrm{d}q_L\mathrm{d}q_R$. Several cases have to be distinguished for the indices $k_1,k_2 \to m_1,m_2,m_3,m_4$; they follow from the transformations (5) and will not be further specified. Again, for completeness, $\rho_{00}^{(0)} = 1$. – Although the strategy how to calculate bin correlation densities of arbitrary order should be clear, we still want to mention three-bin correlations. Two principal cases have to be taken into account: if k_1,k_2,k_3 all belong to the same branch we have

$$\rho_{k_1k_2k_3}^{(j+1)} = (1-\tilde{p})\rho_{m_1m_2m_3}^{(j)} + \tilde{p}\,\overline{q^3}\,\rho_{m_4m_5m_6}^{(j)} \tag{11}$$

with $\overline{q^3} = \overline{q_L^3} = \int p(q_L,q_R)\,q_L^3\,\mathrm{d}q_L\mathrm{d}q_R$ or $\overline{q^3} = \overline{q_R^3}$, respectively, and if k_1,k_2 belong for example to the left branch and k_3 to the right branch we have

$$\rho_{k_1k_2k_3}^{(j+1)} = (1-\tilde{p})\rho_{m_1m_2m_3}^{(j)} + \tilde{p}\,\overline{q_L^2q_R}\,\rho_{m_4m_5}^{(j)}\rho_{m_6}^{(j)} \quad . \tag{12}$$

The first value for the iteration is given by $\rho_{000}^{(0)} = 1$.

2.3 Specific cascade models

The recursion relations (8)-(12) for the bin correlation densities are quite general. Models of hierarchical random branching processes differ in the branching probability and the splitting function, the latter giving rise to different splitting moments $\overline{q_L^m q_R^n}$. In this subsection we concentrate first on two specific cascade models with determinstic branching, the p-model [4] and the α-model [3]. They differ in that the former conserves energy at each branching whereas the latter does not. Both models have been suggested to describe the intermittent energy dissipation of fully developed turbulence; both also serve as simple discrete approximations to multiparticle processes in high energy e^+e^- and hadron-hadron collisions [5]-[7]. Then we discuss the effects of random branching for the case of the p-model. Finally, a model in the context of perturbative QCD parton cascades [17, 18] is analyzed.

2.3.1 p-model

For the p-model [4] the branching probability is equal to $\tilde{p} = 1$; we call this deterministic branching. With every branching the energy splits into two unequal portions governed by the splitting factors $q_L/2 = (1+\alpha)/2$ and $q_R/2 = (1-\alpha)/2$ or vice versa; both possibilities occur with probability 1/2. The splitting function of the p-model then reads:

$$p(q_L, q_R) = \frac{1}{2}\Big(\delta(q_L - (1+\alpha)) + \delta(q_L - (1-\alpha))\Big)\delta(q_L + q_R - 2) \quad . \tag{13}$$

The last δ-function in this expression guarantees that energy is conserved in the splitting process. Note that q_L and q_R are weight factors for the energy densities and not the energies, so that $q_L + q_R = 2$ and not equal to one. The moments of the splitting function which enter in the recursion relations for the correlation densities (8)–(12) now become:

$$\begin{aligned}
\overline{q_L} &= \overline{q_R} = \int_0^2 p(q_L, q_R)\, q_L \,\mathrm{d}q_L \mathrm{d}q_R = 1 \quad , \\
\overline{q_L^2} &= \overline{q_R^2} = 1+\alpha^2 \quad , \qquad \overline{q_L q_R} = 1-\alpha^2 \quad , \\
\overline{q_L^3} &= \overline{q_R^3} = 1+3\alpha^2 \quad , \qquad \overline{q_L^2 q_R} = \overline{q_L q_R^2} = 1-\alpha^2 \quad ;
\end{aligned} \tag{14}$$

they determine the scaling indices of the correlation densities.

From the recursion relation (8) it follows that the bin correlation densities of first order are equal to

$$\rho_{k_1}^{(J)} = 1 \quad , \tag{15}$$

because the first order splitting moments in Eq. (14) are equal to one. This is just another way of saying that on average the energy density at scale J is equal to the energy density at scale $J = 0$. For second order bin correlation densities we find from Eqs. (9), (10) and (14)

$$\rho_{k_1 k_2}^{(j+1)} = \begin{cases} (1+\alpha^2)\rho_{k_1 k_2}^{(j)} & \text{, if } k_1, k_2 \in \{L\} \quad , \\ (1+\alpha^2)\rho_{k_1 - 2^j, k_2 - 2^j}^{(j)} & \text{, if } k_1, k_2 \in \{R\} \quad , \\ (1-\alpha^2) & \text{, if } k_1 \in \{L\} \text{ and } k_2 \in \{R\} \text{ or vice versa.} \end{cases} \tag{16}$$

The power-law-like singularity towards the diagonal can be seen clearly in Fig. 3a. The closer the bins are together topologically the more they share a common history and the stronger they are correlated. This is a characteristic feature of selfsimilarity. We shed some more light onto Eq. (16), once we compare for example $\rho_{00}^{(J)} = \rho_{11}^{(J)} = (1+\alpha^2)^J$ and $\rho_{01}^{(J)} = \rho_{10}^{(J)} = (1+\alpha^2)^{J-1}(1-\alpha^2)$; they differ by a factor $(1+\alpha^2)/(1-\alpha^2)$. This represents an anticorrelation caused by energy conservation: $\epsilon_0^{(j+1)}$ and $\epsilon_1^{(j+1)}$ of one possible configuration differ insofar that only at the very last cascade step different energy-conserving weights $(1+\alpha)$ and $(1-\alpha)$ were assigned to them.

2.3.2 α-model

Like the p-model, the α-model is a deterministic cascade, so that $\tilde{p} = 1$. The splitting function factorizes and is now given as

$$\begin{aligned} p(q_L, q_R) &= \Big(p_1\delta(q_L - (1-\alpha)) + p_2\delta(q_L - (1+\beta))\Big) \\ &\quad \cdot\Big(p_1\delta(q_R - (1-\alpha)) + p_2\delta(q_R - (1+\beta))\Big) \quad , \end{aligned} \tag{17}$$

where the splitting parameters α, β are positive real numbers. The probabilities $p_1 = \beta/(\alpha+\beta)$ and $p_2 = \alpha/(\alpha+\beta)$ are chosen such that $p_1+p_2 = 1$ and $p_1(1-\alpha)+p_2(1+\beta) = 1$. In contrast to the p-model, energy needs not be conserved in every cascade step; for example the energy density may split into a left part with weight factor $(1-\alpha)$ and into a right part with the same weight factor $(1-\alpha)$ with a probability p_1^2. Energy is however conserved on average since $\overline{q_{L/R}} = 1$. For the splitting moments appearing in eqs. (8)-(12) we derive from (17)

$$\begin{aligned} \overline{q_L} &= \overline{q_R} = 1 \quad , \\ \overline{q_L^2} &= \overline{q_R^2} = 1+\alpha\beta \quad , \qquad & \overline{q_L q_R} &= 1 \quad , \\ \overline{q_L^3} &= \overline{q_R^3} = 1+\alpha\beta(3-\alpha+\beta) \quad , \qquad & \overline{q_L^2 q_R} &= \overline{q_L q_R^2} = 1+\alpha\beta \quad . \end{aligned} \tag{18}$$

With the first order splitting moments $\overline{q_L}$, $\overline{q_R}$ and the branching probability $\tilde{p} = 1$ we derive from the recursion relation (8)

$$\rho_{k_1}^{(J)} = 1 \quad . \tag{19}$$

On average the energy density contained in one bin is equal to the original energy density.

According to the recursion relations (9), (10) and the corresponding splitting moments (18) we derive for the second order bin correlations

$$\rho_{k_1k_2}^{(j+1)} = \begin{cases} (1+\alpha\beta)\rho_{k_1k_2}^{(j)} & \text{, if } k_1, k_2 \in \{L\} \text{ or } \ldots \\ 1 & \text{, if } k_1 \in \{L\} \text{ and } k_2 \in \{R\} \text{ or vice versa,} \end{cases} \tag{20}$$

which are visualized in Fig. 3b. Apparently there is no big difference to the corresponding p-model relations (16): A powerlaw scaling towards the diagonal emerges, which is a clear

evidence of the underlying selfsimilarity in the α-model cascade. The difference to the p-model correlations lies in the absence of anticorrelations due to the violation of energy conservation. As soon as two different bins start to belong to different branches during the evolution of the cascade they become completely uncorrelated. We will see later on, that this missing anticorrelation is responsible for the deviation from perfect scaling behaviour of the α-model in a moment (multifractal) analysis.

2.3.3 p-model with random branching

In the previous subsections we have discussed the p- and α-model as two representatives of deterministic branching processes. Although these hierarchical models are intended as simple discrete approximations to multiparticle cascading [5]-[7], they might not be that realistic. Multiparticle branchings have to be somehow "undeterministic": With a certain probability a particle might not decay into two other particles at every step of the cascade; so to say the particle has a certain lifetime. Therefore we incorporate a nonbranching part into the hierarchical cascade process as described in sections 2.1 and 2.2. In fact, the branching rate $1/\tau$ introduced in the evolution equation (7) can be interpreted as the inverse of the particle's lifetime once the scale evolution parameter j is identified with time. With this motivation we study the effect of random branching on selfsimilar cascading.

As in the deterministic p-model, we choose the splitting function (13) with the corresponding splitting moments (14). The branching probability $\tilde{p}$ is now chosen in between $0 < \tilde{p} < 1$; the case $\tilde{p} = 1$ defines deterministic branching and the case $\tilde{p} = 0$ describes no branching at all. The recursion relation (8) for the one-bin correlation density has as solution again $\rho_{k_1}^{(J)} = 1$. The solution of the recursion relations (9) and (10) for the two-bin correlation density is depicted in Fig. 3c; there is an approximate power-law rise far from the diagonal, but the closer the diagonal is approached the more the rise turns into a plateau. The explanation for this behaviour lies in the following: With an increasing number of cascade steps the probability that between neighbouring bins no branching has occurred becomes overwhelming. Then the energy densities of neighbouring bins are the same, so that the two-bin correlation density near the diagonal becomes constant.

2.3.4 QCD-motivated cascade model

As a generalization of the p- and α-model, so-called pseudo QCD-cascade models have been discussed in the context of multiparticle dynamics [20]. Recently Brax, Meunier and Peschanski [17] have suggested a discrete hierarchical branching model, which reproduces the same multiplicity moments as more elaborate gluon jet cascade models within perturbative QCD [17],[21]-[23]. In this scenario the interval splitting at each branching step is interpreted as a partitioning in opening angle. The simplicity of the QCD motivated model of refs. [17, 18] permits a full analytical treatment in terms of (local) correlation densities besides (global) moments.

The branching probability is set to $\tilde{p} = 1$, reflecting the "zero lifetime" of the virtual gluons. The splitting function reads

$$p(q_L, q_R) = \frac{1}{2}\left[\delta(q_L - 2)\left(\delta(q_R) + \left(\frac{\gamma}{q_R}\right)_+\right) + \left(\delta(q_L) + \left(\frac{\gamma}{q_L}\right)_+\right)\delta(q_R - 2)\right] \tag{21}$$

and describes the decay of a hard parent gluon into one hard and one soft daughter gluon. The quantity $(1/q)_+ = \lim_{\beta\to 0}\left((1/q)\Theta(q-2\beta) + \delta(q-2\beta)\ln\beta\right)$ indicates a regularization prescription [24]. The parameter $\gamma = \gamma_0 \ln 2 = \sqrt{6\alpha_s/\pi}\ln 2$ is proportional to the fixed strong coupling constant $\sqrt{\alpha_s}$. With (21) the splitting moments become

$$\begin{aligned}
\overline{q_L} &= \overline{q_R} = \int_0^2 \mathrm{d}q_L \mathrm{d}q_R\, p(q_L, q_R)\, q_L = 1+\gamma \quad , \\
\overline{q_L^2} &= \overline{q_R^2} = 2(1+\frac{\gamma}{2}) \quad , \qquad \overline{q_L q_R} = 2\gamma \quad , \\
\overline{q_L^3} &= \overline{q_R^3} = 4(1+\frac{\gamma}{3}) \quad , \qquad \overline{q_L^2 q_R} = \overline{q_L q_R^2} = 6\gamma \quad .
\end{aligned} \tag{22}$$

This sets the stage for the conventional correlation densities (8)-(12).

For the first order correlation density we find

$$\rho_{k_1}^{(J)} = (1+\gamma)^J \quad . \tag{23}$$

The splitting function (21) does not conserve energy; it produces additional energy in every cascade step. Here, we should think of "energy" as "multiplicity", so that (23) describes the average production of multiplicity during the evolution of the cascade.

With the help of the recursion relations (9) and (10), the relevant splitting moments (22) and the result (23) we calculate for the bin correlation densities of second order

$$\rho_{k_1 k_2}^{(j+1)} = \begin{cases} (2+\gamma)\rho_{k_1 k_2}^{(j)} & \text{, if } k_1, k_2 \in \{L\} \text{ or } \ldots \\ 2\gamma(1+\gamma)^{2j} & \text{, if } k_1 \in \{L\} \text{ and } k_2 \in \{R\} \text{ or vice versa.} \end{cases} \tag{24}$$

The reduced second order correlation density $\rho_{k_1 k_2}/\rho_{k_1}\rho_{k_2}$ is shown in Fig. 3d. Note that for the p- and α-model with and without random branching it was superfluous to distinguish between a reduced and non-reduced correlation density because there $\rho_k = 1$. This is not anymore the case for the QCD-motivated cascade model treated here.

From Eq. (24) it follows with $k_1 = 0$ that $\rho_{0,k}/\rho_{0,2k} = (2+\gamma)/(1+\gamma)^2$ for $k \neq 0$ and $\rho_{0,0}/\rho_{0,1} = (2+\gamma)/2\gamma$. This implies a scaling $[(2+\gamma)/(1+\gamma)^2]^j$ towards the diagonal except for the last step. It is the same scaling we encounter in a (forward) normalized moment analysis in section 2.4.2. Due to the pronounced "non-scaling" peak along the diagonal and the non-matching anticorrelations, the correct (backward) moments will deviate from this perfect scaling.

2.4 Moments: forward evolution vs. backward analysis

Contrary to common belief, the α-model does not exhibit a perfect power law in a (backward) moment analysis in spite of its strictly selfsimilar (forward) evolution. This statement also holds for the QCD-motivated cascade model discussed above. In order to unveil this hidden secret we recall the standard procedure of a moment analysis as proposed in ref. [5].

For example, the second order moment $M_2(J,J)$ is defined by

$$M_2(J,J) = \frac{1}{2^J}\sum_{k=0}^{2^J-1}\left\langle (\epsilon_k^{(J)})^2\right\rangle = \frac{1}{2^J}\sum_{k=0}^{2^J-1}\rho_{kk}^{(J)} = \rho_{00}^{(J)} \quad , \tag{25}$$

where the first index j_1 in the nonstandard, double-indexed notation $M_2(j_1,j_2)$ denotes the total number of cascade steps J after a forward evolution whereas the second index $j_2 \le j_1$ represents the resolution scale ($\sim \log(\text{bin size})$), at which the cascade is analyzed in backward direction. Note that the bin correlation densities $\rho_{kk}^{(J)}$ of the p- and α-model as well as of the QCD-motivated cascade model are independent of the shift index k. Now we average over two neighbouring bins and arrive at the moment

$$M_2(J,J-1) = \frac{1}{2^{J-1}}\sum_{k=0}^{2^{J-1}-1}\left\langle \left(\frac{1}{2}(\epsilon_{2k}^{(J)}+\epsilon_{2k+1}^{(J)})\right)^2\right\rangle = \frac{1}{2}(\rho_{00}^{(J)}+\rho_{01}^{(J)}) = \frac{1}{2}(M_2(J,J)+\rho_{01}^{(J)}) \quad , \tag{26}$$

resolved at the scale $j_2 = J-1$. Then, we continue to average over four, eight, ... neighbouring bins and arrive at the following recursion relation for the moments:

$$M_2(J,J-j-1) = \frac{1}{2}\left(M_2(J,J-j)+\rho_{0,2^j}^{(J)}\right) \quad . \tag{27}$$

2.4.1 p- and α-model

For the p-model we realize with the help of (16) that

$$p\text{-model}: \qquad M_2(J,j) = M_2(j,j) \equiv \rho_{kk}^{(j)} = (1+\alpha^2)^j \quad . \tag{28}$$

The moment obtained after J cascade steps, averaged over bins of size 2^{J-j} and then again averaged over 2^j blocks of bins is equal to the diagonal element of the bin correlation density obtained after j cascade steps.

This does not hold anymore for the α-model: From (27) and (20) we derive

$$\alpha\text{-model}: \qquad M_2(J,j) = \frac{1}{(1-\alpha\beta)}(1+\alpha\beta)^j\left\{1-\alpha\beta\left(\frac{1+\alpha\beta}{2}\right)^{J-j}\right\} \quad . \tag{29}$$

In Fig. 4 we summarize the various scaling aspects of the moments $M_2(J,j)$. Obviously, the backward analyzed moment $M_2(J,j)$ is not equal to the bin correlation density $\rho_{kk}^{(j)} =$

$(1+\alpha\beta)^j = M_2(j,j)$ obtained after j forward α-model cascade steps. The reason for this lies in the fact that energy is not conserved in the α-model; an energy density $\epsilon_k^{(j)}$ obtained after j cascade steps might split into two energy densities $\epsilon_{2k}^{(j+1)} = (1+\beta)\epsilon_k^{(j)}$ and $\epsilon_{2k+1}^{(j+1)} = (1+\beta)\epsilon_k^{(j)}$ in the next cascade step. But then $\epsilon_k^{(j)} \neq \frac{1}{2}(\epsilon_{2k}^{(j+1)} + \epsilon_{2k+1}^{(j+1)})$ so that $M_2(j+1,j) \neq \rho_{kk}^{(j)} = M_2(j,j)$.

Here it is really important to distinguish between the moment analysis in evolution direction of the cascade, i.e. $m_2(j) = M_2(j,j)$ in dependence of j, and in backward direction after J cascade steps, i.e. $M_2(J,j)$ in dependence of j. An experimental moment analysis is always based on moments like $M_2(J,j)$: every configuration (event) is analyzed backwards, given a fully developed cascade at final scale J. From these considerations about the α-model we learn, that the (observed) moments $M_2(J,j)$ may not be naively identified with the forward evolution moments $m_2(j)$. This is in contrast with the results of the p-model cascade, where the anticorrelations due to energy conservation make a distinction between an evolution and a backward analysis unnecessary.

2.4.2 QCD-motivated cascade model

From the (backward) moment recursion relation (27) and Eq. (24) we derive for the QCD-motivated cascade model

$$M_2(J,j) = 2\frac{(1+\gamma)^{2J}}{3+2\gamma}\left(\frac{2+\gamma}{(1+\gamma)^2}\right)^j + \frac{1+2\gamma}{3+2\gamma}\left(1+\frac{\gamma}{2}\right)^J 2^j \quad . \tag{30}$$

This is an unnormalized moment. For a proper normalization we have to divide by the square of the density $(\rho_0^{(J)})^2 = (1+\gamma)^{2J}$ from Eq. (23); this normalization does not affect the scale j dependence.

We observe that the above expression consists of two terms with different scaling behaviour. The first term scales with a factor we have already noticed in the second order correlation densities (24), whereas the second term scales trivially in the factor 2. This can be explained in the following way: The second order correlation density (24) can be decomposed into a correlation matrix with exact anticorrelations and an additional purely diagonal correlation matrix; the former gives rise to the first term in (30) with anomalous scaling, whereas the latter accounts for the trivial scaling in powers of 2 of the second term.

Note in this context, that the expression for the second order moment (29) of the α-model can also be cast in this form: one term scales with the factor $(1+\alpha\beta)$, whereas the second term scales with the factor 2. In contrast to the α-model result, the second term in (30) for the QCD-motivated cascade model comes with a positive sign. As a consequence, in a log-log plot the curve for M_2 bends upwards for the fine resolution scales, i.e. large j values, and deviates from the perfect scaling shown at the rougher resolution scales, i.e. small j values, where the first term in (30) dominates over the second one. This result is illustrated in Fig. 5; compare also again with the corresponding α-model result shown in Fig. 4.

We emphasize again that the (backward) moment $M_2(J,j)$ of (30) is the relevant moment to be compared to an experimental analysis. In theory, the normalized moments $m_q(j)$

in forward evolution of the cascade are calculated in most cases and compared directly to experimental data. For example, the corresponding normalized moment of second order is defined as

$$m_2(j) = \frac{\rho_{kk}^{(j)}}{(\rho_k^{(j)})^2} \quad . \tag{31}$$

Here the diagonal elements of the correlation densities enter for a cascade with j cascade steps; the cascade is *not* evolved down to the finest scale J and then resolved backwards on a rougher resolution scale j. With the results (23) and (24) we find for the QCD-motivated cascade model the scaling law

$$\begin{aligned} m_2(j) &= \left(\frac{2+\gamma}{(1+\gamma)^2}\right)^j \\ &= \left(2^{-j}\right)^{-\log_2(2+\gamma)+2\log_2(1+\gamma)} \\ &\approx \left(2^{-j}\right)^{-(1-\frac{3}{2}\gamma_0)} \qquad (\text{for } \gamma_0 \ll 1). \end{aligned} \tag{32}$$

It shows the same anomalous scaling as the first term in the corresponding (backward) moment (30) and is also illustrated in Fig. 5. The exponent $\frac{3}{2}\gamma_0$ appearing in the last expression is identified with the Renyi dimension D_2.

For the (forward) normalized moments of higher order the calculation is straightforward; we state the outcome without giving further details:

$$\begin{aligned} m_q(j) &= \left(2^{-j}\right)^{-\log_2\left[2^{q-1}(1+\frac{\gamma}{q})\right]+q\log_2(1+\gamma)} \\ &\approx \left(2^{-j}\right)^{-(q-1)(1-D_q)} \qquad (\text{for } \gamma_0 \ll 1) \end{aligned} \tag{33}$$

with

$$D_q = \frac{q+1}{q}\gamma_0 \quad . \tag{34}$$

These scaling exponents for the (forward) evolution moments have been derived previously in the context of a perturbative QCD gluon cascade [17],[21]-[23]. Note again that it is not these unobservable (forward) evolution moments which should be compared to experimental data, but only experimentally accessible (backward) moments.

3 Wavelet correlations in hierarchical branching processes

The conventional correlation densities (1) characterize the correlations between bin contents resolved at some finest scale J (usually chosen somewhere near the resolution limit of the measurement device). They do not provide a natural description in terms of larger structures (clusters and voids). However, sometimes a choice of a clever basis for the representation

of the bin contents can directly unveil this information of interest. The wavelet transform [10]-[14] appears to be such a clever change of basis functions for hierarchically organized stochastic processes [15, 16]. The strategy applied in this section is the following: We apply a multiresolution analysis arising from the wavelet transform to each realization of the branching process and study the correlations between the wavelet amplitudes. This procedure is exemplified for the same cascade models discussed in section 2.

3.1 Multiresolution analysis

In this subsection we will briefly recite only the most important ingredients of a wavelet-based multiresolution analysis. We will concentrate on the Haar wavelet only, which is the simplest of all wavelets; generalizations to other wavelets are straightforward.

The binned energy densities $\epsilon_k^{(J)}$ can be viewed as a step function of $x \in [0,1]$:

$$\epsilon^{(J)}(x) = \sum_{k=0}^{2^J-1} \epsilon_k^{(J)} \phi_{Jk}(x) \quad , \tag{35}$$

where

$$\phi_{Jk}(x) = \phi(2^J x - k) = \begin{cases} 1 & \text{for } k2^{-J} \le x \le (k+1)2^{-J} \\ 0 & \text{else} \end{cases} \quad , \tag{36}$$

are simple box functions. In this respect the energy densities $\epsilon_k^{(J)}$ can be interpreted as amplitudes of the orthogonal expansion of $\epsilon^{(J)}(x)$ in terms of the basis functions $\phi_{Jk}(x)$ at scale J.

The fundamental dilation equations

$$\begin{aligned} \phi_{j-1,k}(x) &= \phi(2^{j-1}x - k) = \sum_m c_m \phi(2^j x - 2k - m) \\ &= \sum_m c_m \phi_{j,2k+m}(x) \quad , \\ \psi_{j-1,k}(x) &= \psi(2^{j-1}x - k) = \sum_m (-1)^m c_{1-m} \phi(2^j x - 2k - m) \\ &= \sum_m (-1)^m c_{1-m} \phi_{j,2k+m}(x) \end{aligned} \tag{37}$$

define a multiresolution analysis [10]-[12]. The associated functions ϕ and ψ are called scaling function and wavelet, respectively. For the box functions (36) we have $c_0 = c_1 = 1$ and $c_k = 0$ otherwise. A different choice for admissible coefficients c_k corresponds to a different filter for the multiresolution analysis and leads to a different wavelet $\psi(x)$; see ref.[11] for a variety of examples.

In Eq. (37) the box functions $\phi_{Jk}(x)$ at scale J are expressed in terms of the box functions $\phi_{J-1,k'}(x)$ at scale $J-1$ and the "difference functions" $\psi_{J-1,k''}(x)$ at scale $J-1$ with

$$\psi(x) = \begin{cases} 1 & \text{for } 0 \le x < \frac{1}{2} \\ -1 & \text{for } \frac{1}{2} \le x < 1 \end{cases} \quad . \tag{38}$$

The set of functions $\psi_{jk}(x)$ is called Haar-wavelet basis. Further iterating Eqs. (37), the energy density step function $\epsilon^{(J)}(x)$ of Eq. (35) can be expressed as a double sum

$$\epsilon^{(J)}(x) = \sum_{j=0}^{J-1} \sum_{k=0}^{2^j-1} \tilde{\epsilon}_{jk}^{(J)} \psi_{jk}(x) + \epsilon_0^{(0)} \phi_{00}(x) \tag{39}$$

representing the multiscale decomposition. In other words, $\epsilon^{(J)}(x)$ is dissected into contributions from different scales j. Eq. (39) defines the linear wavelet transformation, which is governed by the coefficients c_k entering Eq. (37):

$$\vec{\tilde{\epsilon}} = \mathbf{W}\vec{\epsilon} = \mathbf{W}(c_k)\vec{\epsilon} \tag{40}$$

with

$$\begin{aligned} \vec{\epsilon} &= (\epsilon_0,\, \epsilon_1,\, \ldots,\, \epsilon_{2^J-1}) \quad , \\ \vec{\tilde{\epsilon}} &= (\epsilon_0^{(0)},\, \tilde{\epsilon}_{00},\, \tilde{\epsilon}_{10},\, \tilde{\epsilon}_{11},\, \tilde{\epsilon}_{20},\, \ldots,\, \tilde{\epsilon}_{J-1,2^{J-1}-1}) \\ &=: (\tilde{\epsilon}_0,\, \tilde{\epsilon}_1,\, \tilde{\epsilon}_2,\, \ldots,\, \tilde{\epsilon}_{2^J-1}) \quad , \end{aligned} \tag{41}$$

where the upper index (J) has been omitted. The transformed amplitudes $\tilde{\epsilon}_{jk}^{(J)}$ are called wavelet amplitudes. The explicit form of the transformation matrix $\mathbf{W}$ is exhibited for example in ref. [16].

3.2 Wavelet transformed correlation densities

Correlations between the transformed amplitudes $\tilde{\epsilon}_{jk}$ are called wavelet correlations and have been introduced already in previous publications [15, 16]. The wavelet correlation densities follow by applying the wavelet transformation either directly to the conventional correlation densities or to the evolution equation for the corresponding generating function.

If the bin correlations (1) are known, the wavelet correlations can be directly deduced with the help of (40):

$$\begin{aligned} \tilde{\rho}_{k_1} &\equiv \langle \tilde{\epsilon}_{k_1} \rangle = \left\langle \sum_{k_2} \mathbf{W}_{k_1 k_2} \epsilon_{k_2} \right\rangle = \sum_{k_2} \mathbf{W}_{k_1 k_2} \langle \epsilon_{k_2} \rangle = \sum_{k_2} \mathbf{W}_{k_1 k_2} \rho_{k_2} \quad , \\ \tilde{\rho}_{k_1 k_3} &\equiv \langle \tilde{\epsilon}_{k_1} \tilde{\epsilon}_{k_3} \rangle = \sum_{k_2 k_4} \mathbf{W}_{k_1 k_2} \mathbf{W}_{k_3 k_4} \rho_{k_2 k_4} \quad , \\ \tilde{\rho}_{k_1 k_3 k_5} &\equiv \langle \tilde{\epsilon}_{k_1} \tilde{\epsilon}_{k_3} \tilde{\epsilon}_{k_5} \rangle = \sum_{k_2 k_4 k_6} \mathbf{W}_{k_1 k_2} \mathbf{W}_{k_3 k_4} \mathbf{W}_{k_5 k_6} \rho_{k_2 k_4 k_6} \quad . \end{aligned} \tag{42}$$

Thus the wavelet correlations can be obtained directly from the standard bin correlations. The treatment of wavelet correlations is general and does not depend on the specific choice of admissible coefficients c_k defining different wavelets.

For the case of the simple Haar wavelet it is instructive to deduce the wavelet correlations directly from the generating function (2). This gives more analytical insight into the correlation structure especially for the binary branching processes we have introduced in section

2. In the exponent of the generating function (2) we introduce the change into the Haar wavelet basis such that

$$\begin{aligned}\sum_{k_1=0}^{2^J-1}\lambda_{k_1}\epsilon_{k_1} &= \sum_{k_1,k_3=0}^{2^J-1}\lambda_{k_1}\delta_{k_1k_3}\epsilon_{k_3} = \sum_{k_1,k_2,k_3=0}^{2^J-1}\lambda_{k_1}(\mathbf{W^{-1}})_{k_1k_2}(\mathbf{W})_{k_2k_3}\epsilon_{k_3} \\ &= \sum_{k_1,k_2=0}^{2^J-1}(\mathbf{W^{-1}})_{k_1k_2}\lambda_{k_1}\tilde{\epsilon}_{k_2} \equiv \sum_{k_2=0}^{2^J-1}\eta_{k_2}\tilde{\epsilon}_{k_2} \quad . \end{aligned} \tag{43}$$

The coordinates η_k show a simple relation with respect to the λ_k:

$$\begin{aligned}\eta_0 &= \sum_{k=0}^{2^J-1}\lambda_k \quad , \\ \eta_1 &= \eta_{00} = \sum_{k=0}^{2^{J-1}-1}\lambda_k - \sum_{k=2^{J-1}}^{2^J-1}\lambda_k \quad , \\ \eta_2 &= \eta_{10} = \sum_{k=0}^{2^{J-2}-1}\lambda_k - \sum_{k=2^{J-2}}^{2^{J-1}-1}\lambda_k \quad , \qquad \eta_3 = \eta_{11} = \sum_{k=2^{J-1}}^{3\cdot 2^{J-2}-1}\lambda_k - \sum_{k=3\cdot 2^{J-2}}^{2^J-1}\lambda_k \quad , \\ &\vdots \\ \eta_{2^{J-1}} &= \eta_{J-1,0} = \lambda_0 - \lambda_1 \quad , \qquad \ldots, \quad \eta_{2^J-1} = \eta_{J-1,2^{J-1}-1} = \lambda_{2^J-2} - \lambda_{2^J-1} \quad ; \end{aligned} \tag{44}$$

for more details see again ref. [16]. In view of the evolution equation (4) with the splitting rules (5) it is necessary to express $\vec{\eta}^{(j+1)}$ at scale $j+1$ in terms of $\vec{\eta}_L^{(j)}$, $\vec{\eta}_R^{(j)}$ and $\vec{\eta}_M^{(j)}$ at scale j; we deduce:

$$\begin{aligned}\vec{\eta}^{(j+1)} &= \left(\eta_0^{(j+1)},\, \eta_{00}^{(j+1)},\, \eta_{10}^{(j+1)},\, \eta_{11}^{(j+1)},\, \eta_{20}^{(j+1)},\, \ldots,\, \eta_{j,2^j-1}^{(j+1)}\right) \quad , \\ \vec{\eta}_L^{(j)} &= \left(\frac{1}{2}(\eta_0^{(j+1)} + \eta_{00}^{(j+1)}),\, \eta_{10}^{(j+1)},\, \eta_{20}^{(j+1)},\, \eta_{21}^{(j+1)},\, \eta_{30}^{(j+1)},\, \ldots,\, \eta_{j,2^{j-1}-1}^{(j+1)}\right) \\ &=: \left(\eta_{0;L}^{(j)},\, \eta_{00;L}^{(j)},\, \eta_{10;L}^{(j)},\, \eta_{11;L}^{(j)},\, \eta_{20;L}^{(j)},\, \ldots,\, \eta_{j-1,2^{j-1}-1;L}^{(j)}\right) \quad , \\ \vec{\eta}_R^{(j)} &= \left(\frac{1}{2}(\eta_0^{(j+1)} - \eta_{00}^{(j+1)}),\, \eta_{11}^{(j+1)},\, \eta_{22}^{(j+1)},\, \eta_{23}^{(j+1)},\, \eta_{34}^{(j+1)},\, \ldots,\, \eta_{j,2^j-1}^{(j+1)}\right) \\ &=: \left(\eta_{0;R}^{(j)},\, \eta_{00;R}^{(j)},\, \eta_{10;R}^{(j)},\, \eta_{11;R}^{(j)},\, \eta_{20;R}^{(j)},\, \ldots,\, \eta_{j-1,2^{j-1}-1;R}^{(j)}\right) \quad , \\ \vec{\eta}_M^{(j)} &= \left(\eta_0^{(j+1)},\, \eta_{00}^{(j+1)},\, \eta_{10}^{(j+1)},\, \eta_{11}^{(j+1)},\, \eta_{20}^{(j+1)},\, \ldots,\, \eta_{j-1,2^{j-1}-1}^{(j+1)}\right) \\ &=: \left(\eta_{0;M}^{(j)},\, \eta_{00;M}^{(j)},\, \eta_{10;M}^{(j)},\, \eta_{11;M}^{(j)},\, \eta_{20;M}^{(j)},\, \ldots,\, \eta_{j-1,2^{j-1}-1;M}^{(j)}\right) \quad . \end{aligned} \tag{45}$$

This sets the stage for the evolution equation of the generating function for the Haar wavelet correlations; from Eq. (4) we deduce straightforwardly

$$\begin{aligned}&Z^{(j+1)}\left[\vec{\eta}^{(j+1)}\right] = \\ &\quad (1-\tilde{p})Z^{(j)}\left[\vec{\eta}_M^{(j)}\right] + \tilde{p}\int \mathrm{d}q_L \mathrm{d}q_R\, p(q_L,q_R)\, Z^{(j)}\left[q_L\vec{\eta}_L^{(j)}\right] Z^{(j)}\left[q_R\vec{\eta}_R^{(j)}\right] \quad . \end{aligned} \tag{46}$$

Taking the derivatives with respect to the $\eta_{j_1 k_1}^{(j+1)}$, recursion relations for the Haar wavelet correlations can be found. For the first order Haar wavelet correlations we find

$$\begin{aligned}
\tilde{\rho}_0^{(j+1)} &= \frac{1}{i}\frac{\partial Z^{(j+1)}[\vec{\eta}^{(j+1)}]}{\partial \eta_0^{(j+1)}}\bigg|_{\vec{\eta}^{(j+1)}=0} = (1-\tilde{p})\tilde{\rho}_0^{(j)} + \tilde{p}\frac{\overline{q_L}+\overline{q_R}}{2}\tilde{\rho}_0^{(j)} \quad , \\
\tilde{\rho}_{(00)}^{(j+1)} &= (1-\tilde{p})(1-\delta_{j0})\tilde{\rho}_{(00)}^{(j)} + \tilde{p}\frac{\overline{q_L}-\overline{q_R}}{2}\tilde{\rho}_0^{(j)} \quad , \\
\tilde{\rho}_{(j_1 k_1)}^{(j+1)} &= (1-\tilde{p})(1-\delta_{jj_1})\tilde{\rho}_{(j_1 k_1)}^{(j)} + \tilde{p}\,\overline{q}\,\tilde{\rho}_{(j_1-1,k_2)}^{(j)} \quad ; \qquad (47)
\end{aligned}$$

the last relation holds for $1 \leq j_1 \leq j$ and $\overline{q} = \overline{q_L}$, $k_2 = k_1$ for $(j_1 k_1) \in \{L\}$ or $\overline{q} = \overline{q_R}$, $k_2 = k_1 - 2^{j_1-1}$ for $(j_1 k_1) \in \{R\}$. The starting value is $\tilde{\rho}_0^{(0)} = \rho_0^{(0)} = 1$. Furthermore, we specify three recursion relations for the second order Haar wavelet correlations, which will turn out to be important for later discussions:

$$\begin{aligned}
\tilde{\rho}_{0,0}^{(j+1)} &= \frac{1}{i^2}\frac{\partial^2 Z^{(j+1)}[\vec{\eta}^{(j+1)}]}{\partial (\eta_0^{(j+1)})^2}\bigg|_{\vec{\eta}^{(j+1)}=0} \\
&= (1-\tilde{p})\tilde{\rho}_{0,0}^{(j)} + \tilde{p}\frac{(\overline{q_L^2}+\overline{q_R^2})}{4}\tilde{\rho}_{0,0}^{(j)} + \tilde{p}\frac{\overline{q_L q_R}}{2}(\tilde{\rho}_0^{(j)})^2 \quad , \\
\tilde{\rho}_{(00),(00)}^{(j+1)} &= (1-\tilde{p})(1-\delta_{j0})\tilde{\rho}_{(00),(00)}^{(j)} + \tilde{p}\frac{(\overline{q_L^2}+\overline{q_R^2})}{4}\tilde{\rho}_{0,0}^{(j)} - \tilde{p}\frac{\overline{q_L q_R}}{2}(\tilde{\rho}_0^{(j)})^2 \quad , \\
\tilde{\rho}_{(j_1 k_1),(j_1 k_1)}^{(j+1)} &= (1-\tilde{p})(1-\delta_{jj_1})\tilde{\rho}_{(j_1 k_1),(j_1 k_1)}^{(j)} + \tilde{p}\,\overline{q^2}\,\tilde{\rho}_{(j_1-1,k_2),(j_1-1,k_2)}^{(j)} \quad , \qquad (48)
\end{aligned}$$

where again $\overline{q^2} = \overline{q_L^2}$ or $\overline{q_R^2}$ and $k_2 = k_1$ or $k_1 - 2^{j_1-1}$ for $(j_1 k_1) \in \{L\}$ or $\{R\}$ in the last relation.

3.3 Wavelet correlations of specific cascade models

3.3.1 p-model

Wavelet correlations in the p-model were already studied in ref. [16], emphasizing the self-similarity aspect of the wavelet basis. We briefly recall the main results. Since the splitting moments $\overline{q_L}$ and $\overline{q_R}$ are equal (see Eq. (14)), the first order Haar wavelet correlations (47) are all zero,

$$\tilde{\rho}_{(jk)} = 0 \quad , \qquad (49)$$

except for $\tilde{\rho}_0 = 1$. On average a difference between energy densities of neighbouring bins is zero, as a difference might come with a positive or negative sign with equal probability. Only the global average $\tilde{\rho}_0$ of all bins is one.

From (48) and (14) we deduce for the second order Haar wavelet correlations

$$\begin{aligned}
\tilde{\rho}_{0,0} &= 1 \quad , \\
\tilde{\rho}_{(j_1 k_1),(j_2 k_2)} &= \alpha^2(1+\alpha^2)^{j_1}\delta_{j_1 j_2}\delta_{k_1 k_2} \quad ; \qquad (50)
\end{aligned}$$

they are depicted in Fig. 6a, where the wavelet indices (jk) are ordered according to Eq. (41). The product of two differences belonging to different scales or bin positions is zero on average. The diagonal contributions, which represent squares of differences, show a scaling law as the resolution j_1 increases; compare with (28). It is exactly this diagonal structure we would have expected from second order Haar wavelet correlations of the selfsimilar p-model cascade. In ref. [16] we have shown that when choosing wavelets other than the Haar wavelet, the second order wavelet correlations of the p-model cascade turn out to be quasidiagonal, with diagonal contributions dominating over the off-diagonal ones and also showing an approximate scaling law.

For the Haar wavelet correlations of the third order a double scaling is found:

$$\tilde{\rho}_{(j_1k_1),(j_2k_2),(j_2k_2)} = (1+3\alpha^2)^{j_1}\left(\pm 2\alpha^4(1+\alpha^2)^{j_2-j_1-1}\right) \quad , \tag{51}$$

where (j_1k_1), (j_2k_2) have to share a common parenthood, i.e. $0 \le j_1 < j_2$, $k_1 2^{j_2-j_1} \le k_2 \le k_1 2^{j_2-j_1} + 2^{j_2-j_1-1} - 1$ for the $(+)$ sign and $k_1 2^{j_2-j_1} + 2^{j_2-j_1-1} \le k_2 \le (k_1+1)2^{j_2-j_1} - 1$ for the $(-)$ sign. All the other pure third order Haar wavelet correlations are zero; only some of those involving an index 0, which represents the amplitude $\tilde{\epsilon}_0 = 1$ of the global average and thus not a difference amplitude, are nonzero but would vanish for Haar wavelet cumulants [16].

Note that the wavelet transform compresses the full information contained in the second order correlation function (16) into the diagonal (50); compare also Fig. 4a with Fig. 6a. In addition to the diagonal contributions also certain off-diagonal bandstructures arise in higher order wavelet correlations as e.g. in Eq. (51). An interpretation of these offdiagonal contributions in terms of clump correlations is given in section 3.4.

3.3.2 α-model

From Eqs. (47) and (18) we derive exactly the same result for the first order Haar wavelet correlations of the α-model as stated in eq. (49) for the p-model. Differences between the two arise for higher order Haar wavelet correlations. With the branching probability $\tilde{p} = 1$ and the second order splitting moments of (18), we derive

$$\begin{aligned} \tilde{\rho}^{(J)}_{0,0} &= \frac{1}{(1-\alpha\beta)}\left\{1-\alpha\beta\left(\frac{1+\alpha\beta}{2}\right)^J\right\} \quad , \\ \tilde{\rho}^{(J)}_{(j_1k_1),(j_2k_2)} &= \frac{\alpha\beta(1+\alpha\beta)^{j_1}}{(1-\alpha\beta)}\left\{1-\left(\frac{1+\alpha\beta}{2}\right)^{J-j_1}\right\}\delta_{j_1j_2}\delta_{k_1k_2} \end{aligned} \tag{52}$$

from the recursion relations (48) for the second order Haar wavelet correlations. This correlation matrix, which is depicted in Fig. 6b, is diagonal as in the p-model case. However, the diagonal contributions now depend on the number of performed cascade steps J.

We first comment on the element $\tilde{\rho}^{(J)}_{0,0}$. In contrast to the p-model result, it is unequal to one; for $|\alpha\beta| < 1$ it is equal to $1/(1-\alpha\beta)$ in the limit $J \to \infty$. The quantity $\tilde{\rho}^{(J)}_{0,0} - 1$ reflects

the square width of the fluctuation in the total energy. Its deviation from zero is a clear evidence that the total energy is not conserved in the α-model cascade.

The elements $\tilde{\rho}^{(J)}_{(j_1k_1),(j_1k_1)}$, reflecting the true second order Haar wavelet correlations, do not show a rigorous scaling; only for j_1 values being "safely" smaller than the number of cascade steps J and $|\alpha\beta| < 1$ does a power law depending on the scale index j_1 show up. If we set $\alpha = \beta$ in the α-model and use the same value as in the p-model, we obtain the same asymptotic scaling as for the perfect scaling in the p-model. The only difference is that the missing anticorrelations lead to a modification of $1/(1-\alpha\beta)$ in the overall factor.

In Fig. 4, we show the deviation from scaling of the diagonal elements of the second order Haar wavelet transformed correlations $\tilde{\rho}^{(J)}_{(jk),(jk)}$ with respect to the scale index j for both the α- and the p-model. Note the resemblance between Eq. (52) and the moment Eq. (29), which emphasizes again that the wavelet correlations imply, by design, a backward analysis. Note also by comparison of Figs. 3b and 6b that these deviations from perfect scaling are observed more easily in the wavelet picture.

3.3.3 p-model with random branching

We do not learn anything new from the recursion relations (47) for the first order Haar wavelet correlations. The global average is $\tilde{\rho}^{(J)}_0 = 1$; first order differences vanish for any $\tilde{p}$, so that $\tilde{\rho}^{(J)}_{(jk)} = 0$.

Turning to the more interesting second order Haar wavelet correlations (48), we find, as expected, that $\tilde{\rho}^{(J)}_{0,0} = 1$. Again, energy is conserved in every possible configuration. Off-diagonal contributions to the second order Haar wavelet correlation matrix $\tilde{\rho}^{(J)}_{(j_1k_1),(j_2k_2)}$ vanish also in this more general cascade. Next, we consider the two diagonal elements $\tilde{\rho}^{(J)}_{(0,0),(0,0)}$ and $\tilde{\rho}^{(J)}_{(J-1,k),(J-1,k)}$ in particular.

For $\tilde{\rho}^{(J)}_{(0,0),(0,0)}$ we find

$$\tilde{\rho}^{(J)}_{(0,0),(0,0)} = \alpha^2 \left(1 - (1-\tilde{p})^J\right) \quad . \tag{53}$$

The factor $(1-\tilde{p})^J$ represents the probability that after J cascade steps no branching has occurred. For $0 < \tilde{p} < 1$ and $J \to \infty$ this probability goes to zero. Then $\tilde{\rho}^{(J)}_{(0,0),(0,0)}$, which represents the square of the difference between the average of the energy densities belonging to the 2^{J-1} left bins and the average of those belonging to the 2^{J-1} right bins, approaches the value (50) obtained for the p-model with $\tilde{p} = 1$.

The element $\tilde{\rho}^{(J)}_{(J-1,k),(J-1,k)}$ representing the square of the difference between energy densities in two adjacent bins at the finest scale follows from the third equation of (48):

$$\tilde{\rho}^{(J)}_{(J-1,k),(J-1,k)} = \tilde{p}\alpha^2 \left(\tilde{p}(1+\alpha^2)\right)^{J-1} \quad . \tag{54}$$

For $\tilde{p} < 1/(1+\alpha^2)$ this element tends to zero as $J \to \infty$. As the number of cascade steps increases, the probability that no branching has occurred on the finest scales also increases, so

that the difference amplitudes on the finest scales will be zero. Once $\tilde{p} \geq 1/(1+\alpha^2)$ however, this interpretation no longer holds. For $\tilde{p} = 1/(1+\alpha^2)$ we have always $\tilde{\rho}^{(J)}_{(J-1,k),(J-1,k)} = \tilde{p}\alpha^2$, no matter how large the number of cascade steps J. Somehow this seems to contradict common sense, especially when the number of cascade steps becomes very large. It is true that, as $J \to \infty$, the probability of a branching occurring at every cascade step becomes infinitesimal small; but on the other hand, once it happens, the difference amplitudes on the finest scales become very large because neighbouring bins have gone through a long common history and have built up huge energy densities. Differences of large densities are also large since the splitting in the very last cascade step always goes with a $(1+\alpha)$ to the left and a $(1-\alpha)$ to the right or vice versa. This huge difference then dominates over the small probability that branching occurs; as a consequence the correlation $\tilde{\rho}^{(J)}_{(J-1,k),(J-1,k)}$ stays finite and unequal zero for $\tilde{p} = 1/(1+\alpha^2)$. For $\tilde{p} > 1/(1+\alpha^2)$ it even increases as the number of cascade steps J goes to infinity.

Fig. 6c shows the full second order Haar wavelet correlation density for the p-model with random branching for the case $\tilde{p} < 1/(1+\alpha^2)$. In Fig. 7, the scale dependence of $\tilde{\rho}^{(J)}_{(j,k),(j,k)}$ is shown for various values of the branching probability $\tilde{p}$. For j much smaller than J the same scaling shows up for the p-model with random branching as with deterministic branching; large-scale branchings have occurred during the cascade with a probability almost equal to one. For small branching probabilities, i.e. $\tilde{p} < 1/(1+\alpha^2)$, a deviation from scaling sets in at midscales and the contributions $\tilde{\rho}^{(J)}_{(j,k),(j,k)}$ drop rapidly to zero as finer and finer scales are considered. This is clearly a lifetime effect, because branchings at the finest scales will hardly occur.

For higher order Haar wavelet correlations similar modifications are to be expected. The multiple scalings (51) and (60) for higher order Haar wavelet correlations will only live on the larger scales and will break down at the finest scales.

3.3.4 QCD-motivated cascade model

The first order Haar wavelet correlation densities for the QCD-motivated cascade model follow from the recursion relation (47) using the splitting moments (22) and $\tilde{p} = 1$:

$$\begin{aligned} \tilde{\rho}^{(J)}_0 &= (1+\gamma)^J \quad , \\ \tilde{\rho}^{(J)}_{(jk)} &= 0 \quad . \end{aligned} \tag{55}$$

As expected, the global average multiplicity density $\tilde{\rho}^{(J)}_0$ is equal to the local average density $\rho^{(J)}_k$ of Eq. (23), whereas all local differences are zero on average.

From (48) and (22) we deduce for the second order Haar wavelet correlations

$$\tilde{\rho}^{(J)}_{0,0} = \frac{(1+\gamma)^{2J}}{\left(\frac{3}{2}+\gamma\right)} + \frac{\left(\frac{1}{2}+\gamma\right)}{\left(\frac{3}{2}+\gamma\right)}\left(1+\frac{\gamma}{2}\right)^J \quad ,$$

$$\tilde{\rho}^{(J)}_{(j_1k_1),(j_2k_2)} = \left[\frac{(1-\gamma-\gamma^2)}{\left(\frac{3}{2}+\gamma\right)}(1+\gamma)^{2J-2}\left(\frac{2+\gamma}{(1+\gamma)^2}\right)^j + \frac{\left(\frac{1}{2}+\gamma\right)}{\left(\frac{3}{2}+\gamma\right)}\left(1+\frac{\gamma}{2}\right)^J 2^j\right]\delta_{j_1j_2}\delta_{k_1k_2} \quad . \tag{56}$$

The average square of the total multiplicity density $\tilde{\rho}^{(J)}_{0,0}$ is unequal to $(\tilde{\rho}^{(J)}_0)^2$, signaling the absence of proper anticorrelations. The true second order Haar wavelet correlations $\tilde{\rho}^{(J)}_{(j_1k_1),(j_2k_2)}$ are again diagonal; normalized with respect to $(\tilde{\rho}^{(J)}_0)^2$ they are shown in Fig. 6d. The diagonal contributions depend on the number of performed cascade steps J and do not show rigorous scaling. In fact, as in the case of (backward) moments (30), the dependence on the resolution scale j splits into a term with the same anomalous scaling and another term scaling trivially in powers of 2. This j-dependence is once more illustrated in Fig. 5; the deviation from the anomalous scaling at the rougher resolution scales sets in earlier for the wavelet correlations than for the (backward) moments.

It is instructive to introduce wavelet moments $W_q(J,j)$. For the case of second order we define

$$W_2(J,j) = \frac{1}{2^j}\sum_{k=0}^{2^j-1}\tilde{\rho}^{(J)}_{(jk),(jk)} = \tilde{\rho}^{(J)}_{(j0),(j0)} \quad . \tag{57}$$

They are related to the (backward) moments $M_2(J,j)$ of (27) via

$$\begin{aligned} W_2(J,J-j-1) &= M_2(J,J-j) - M_2(J,J-j-1) \\ &= \frac{1}{2}\left(M_2(J,J-j) - \rho^{(J)}_{0,2^j}\right) \quad , \end{aligned} \tag{58}$$

which follows from Eq. (42). This relation reflects the multiresolution property of the wavelets as they look on the difference of two adjacent scales. A "difference" moment is more sensitive to deviations from perfect scaling than an "average" moment; this explains the results depicted in Fig. 5. Also, in view of the relation (58), it now becomes clear that the expression (56) for the wavelet correlations splits into the same two different scaling terms as the (backward) moments (30) did.

Expressions for higher order wavelet correlation densities will not be given here for the QCD-motivated cascade model. Their general interpretation will be given in the next two sections 3.4 and 3.5. For a gluon cascade they can be understood as correlations between gluon subjets inside larger gluon jets.

3.4 Higher order wavelet correlations: clumps

What does the double scaling of the third order wavelet correlation (51) tell us? To get a glimpse, we consider once again the evolution of the energy densities according to the cascading prescription of the p-model. See Fig. 8. After j_1 cascade steps we pick out the energy density $\epsilon^{(j_1)}_{k_1}$ of one bin with label k_1. At the next cascade step this bin has split

into two subbins with labels $2k_1$ and $2k_1+1$ respectively, which contain the energy densities $\epsilon_{2k_1}^{(j_1+1)}$ and $\epsilon_{2k_1+1}^{(j_1+1)}$. We ask now for the correlation between the energy density $\epsilon_{k_2}^{(j_2)}$ contained in a subbin with labels j_2, k_2 and the energy density $\epsilon_{k_1}^{(j_1)}$ contained in the picked bin with labels j_1, k_1 where the support of the subbin j_2, k_2 is contained in the support of the bin j_1, k_1; i.e. $j_1 < j_2$ and $k_1 2^{j_2-j_1} \leq k_2 \leq (k_1+1)2^{j_2-j_1} - 1$. We get for example:

$$\begin{aligned} \left\langle \epsilon_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)} \right\rangle &= \left\langle (\epsilon_{k_1}^{(j_1)})^2 \right\rangle \left\langle \epsilon_{k_3}^{(j_2-j_1)} \right\rangle = (1+\alpha^2)^{j_1} \quad , \\ \left\langle \epsilon_{k_1}^{(j_1)} (\epsilon_{k_2}^{(j_2)})^2 \right\rangle &= \left\langle (\epsilon_{k_1}^{(j_1)})^3 \right\rangle \left\langle (\epsilon_{k_3}^{(j_2-j_1)})^2 \right\rangle = (1+3\alpha^2)^{j_1} (1+\alpha^2)^{j_2-j_1} \quad , \end{aligned} \tag{59}$$

where $k_3 = k_2 - k_1 2^{j_2-j_1}$. These interscale correlations contain additional information about the subclustering structure of the cascade process. In other words, this can be interpreted as the correlation between a "coherent structure" and one of its "substructures". Such (sub)structures can be identified with clusters (high density regions) as well as voids (low density regions) and will subsequently be referred to as "clumps".

The second order correlation $\langle \epsilon_{k_1}^{(j_1)} \epsilon_{k_2}^{(j_2)} \rangle$ only provides information about the common "history" of the clumps $j_1 k_1$ and $j_2 k_2$ and not about the subclustering structure, because $\langle \epsilon_{k_3}^{(j_2-j_1)} \rangle = 1$. This situation changes once we consider the third order correlation $\langle \epsilon_{k_1}^{(j_1)} (\epsilon_{k_2}^{(j_2)})^2 \rangle$. Here we get a double scaling, which depends on the common "history" of the two clumps as well as on the subclustering structure of the larger clump. It is the same double scaling we have found for the third order Haar wavelet correlations (51). Therefore we interpret higher order Haar wavelet correlations as clump correlations. In fact, what the third order Haar wavelet correlations do, is that they correlate the difference of the energy densities of two adjacent clumps living on the scale j_1 with the square of the difference of the energy densities of two neighbouring subclumps living on the scale j_2 inside the original clump, as depicted in Fig. 8 by marked arrows. Of course, this is just another definition of clump correlations.

For completeness we state the nonvanishing contributions to the fourth order Haar wavelet correlations of the p-model which provide further information about the subclumping structure:

$$\begin{aligned} \tilde{\rho}_{(j_1k_1)^4} &= \alpha^4 (1+6\alpha^2+\alpha^4)^{j_1} \ , \\ \tilde{\rho}_{(j_1k_1)^2,(j_2k_2)^2} &= \alpha^4 (1+\alpha^2)^{j_1-j_2} (1+6\alpha^2+\alpha^4)^{j_2} \ , \\ &\qquad (j_2 < j_1, \text{ same parenthood}), \\ &= \alpha^4 (1-\alpha^2)^2 (1+6\alpha^2+\alpha^4)^{s} (1+\alpha^2)^{j_1+j_2-2s-2} \ , \\ &\qquad (\text{partially same parenthood}), \\ \tilde{\rho}_{(j_1k_1)^2,(j_2k_2),(j_3k_3)} &= (\pm)_{j_3} (\pm)_{j_2} 2\alpha^6 (3+\alpha^2)(1+6\alpha^2+\alpha^4)^{j_3} (1+3\alpha^2)^{j_2-j_3-1} \cdot \\ &\quad \cdot (1+\alpha^2)^{j_1-j_2-1} \ , \\ &\qquad (j_3 < j_2 < j_1, \text{ same parenthood}). \end{aligned} \tag{60}$$

The phrase "same parenthood" used in the first case of $\tilde{\rho}_{(j_1k_1)^2,(j_2k_2)^2}$ stands for $k_2 2^{j_1-j_2} \leq k_1 \leq (k_2+1)2^{j_1-j_2} - 1$. The second case with "partially same parenthood" translates into $0 \leq k \leq 2^s - 1$, $k \cdot 2^{j_1-s} \leq k_1 \leq (k+\frac{1}{2})2^{j_1-s} - 1$, $(k+\frac{1}{2})2^{j_2-s} \leq k_2 \leq (k+1)2^{j_2-s} - 1$, where

s represents the scale, from which point the wavelet indices (j_1k_1) and (j_2k_2) follow different branches in the underlying tree structure. $(j_1 - s)$ and $(j_2 - s)$ are the scales of two different subclumps relative to the scale s of their common parent clump. Thus the correlation density $\tilde{\rho}_{(j_1k_1)^2,(j_2k_2)^2}$ provides information not only about the direct subclustering of a large clump, but also relates two different subclumps within a large clump.

Furthermore, the triple scaling of $\tilde{\rho}_{(j_1k_1)^2,(j_2k_2),(j_3k_3)}$ tells us something about the correlation of a subsubclump within a subclump within a clump. Here the phrase "same parenthood" illustrates $k_3 \cdot 2^{j_2-j_3} \leq k_2 \leq (k_3 + \frac{1}{2})2^{j_2-j_3} - 1$ for the $(+)$ sign and $(k_3 + \frac{1}{2})2^{j_2-j_3} \leq k_2 \leq (k_3+1)2^{j_2-j_3}-1$ for the $(-)$ sign in $(\pm)_{j_3}$ and, furthermore, $k_2 \cdot 2^{j_1-j_2} \leq k_1 \leq (k_2+\frac{1}{2})2^{j_1-j_2}-1$ for the $(+)$ sign and $(k_2 + \frac{1}{2})2^{j_1-j_2} \leq k_1 \leq (k_2+1)2^{j_1-j_2} - 1$ for the $(-)$ sign in $(\pm)_{j_2}$. – This demonstrates once more that higher order wavelet correlations provide direct information about the clustering hierarchy and can be interpreted as clump correlations. In the next section we further illuminate this interpretation from a two-dimensional perspective.

We emphasize that this kind of subclustering information cannot be obtained by a conventional moment analysis along the lines of section 2.4. Although in principle this information is contained in the conventional higher-order correlation densities, it is very hard to measure or visualize directly. Due to its multiresolution character, the wavelet transform compresses exactly this information of interest to a readily accessible form.

3.5 Multiscale clustering in two-dimensional branching processes

To illustrate and visualize clumps in more detail, we will now consider hierarchical branching processes in two dimensions. As a representative we choose the two-dimensional α-model. Compared to the p-model, the α-model is easier to generalize to higher dimensions, as it can be built by direct products of one-dimensional α-models. Instead of halving a given interval, a square is subdivided into four subsquares with labels 1,...,4 during one cascade step. The original energy is partitioned according to the splitting function

$$p(q_1, \ldots, q_4) = \prod_{i=1}^{4} [p_1\delta(q_i - (1-\alpha)) + p_2\delta(q_i - (1+\beta))] \quad ; \tag{61}$$

this is a straightforward generalisation of eq. (17). The prescription (61) is repeated for all following cascade steps. One possible realization of the two-dimensional α-model is depicted in the upper left corner of Fig. 9; six cascade steps ($j = 6$) have been performed with the splitting parameters $\alpha = \beta = 0.4$ We have used a gray scale to indicate the population of regions in between large energy densities (white) and small energy densities (black). We observe that certain regions clump into clusters of large densities and other regions with small energy densities exist as voids. Furthermore clusters/voids appear in different sizes and show substructures, which are again clumped into clusters and voids.

To quantify this picture, we explicitly perform a multiresolution analysis: First the energy densities of four little squares constituting a larger square are averaged. These averaged

energy densities are depicted in Fig. 9 as the second figure from the top of the left column. In other words, the original configuration has been smoothed or resolved on a rougher scale j=5. Some detail is obviously lost in this representation; no information about substructures of clumps living on the scale $j = 5$ can be deduced. This lost information can be recovered by keeping the difference between the resolutions j=5 and j=6; this difference information is illustrated in the picture to the right of the j=5 resolution picture. If this difference vanishes in some regions, no substructures are present; if on the other side those differences become large in other regions, then this indicates sizeable substructures on the scale j=6.

These smoothing and differentiation operations are iterated through scales j=4, 3 down to 2. In general, the difference between two successive smoothings of a configuration gives the information about the subclustering present at the involved scales. It is important to notice that the difference information on one scale is completely independent of (orthogonal to) the difference information on any other scale and therefore does not carry along redundant information. Also, no information is lost by keeping only the differences of smoothed representations. The sum of all difference representations together with the roughest smoothed configuration (j=2) recovers the original configuration belonging to the finest resolution scale. This is the two-dimensional analogue to Eq. (39).

The smoothing operations at scale j are performed with the basis

$$\phi_{j,k_1k_2}(x,y) = \phi(2^{-j}x - k_1, 2^{-j}y - k_2) = \phi(2^{-j}x - k_1)\phi(2^{-j}y - k_2) = \phi_{jk_1}(x)\phi_{jk_2}(y) \quad ; \quad (62)$$

this is a direct product of two one-dimensional scaling functions given in eq. (36) as box functions. The difference between two successive smoothing operations can be completely expressed in terms of two-dimensional wavelets, namely

$$\begin{aligned} \psi^{(1)}_{j,k_1k_2}(x,y) &= \phi_{jk_1}(x)\psi_{jk_2}(y) \quad , \\ \psi^{(2)}_{j,k_1k_2}(x,y) &= \psi_{jk_1}(x)\phi_{jk_2}(y) \quad , \\ \psi^{(3)}_{j,k_1k_2}(x,y) &= \psi_{jk_1}(x)\psi_{jk_2}(y) \quad . \end{aligned} \qquad (63)$$

Here ψ_{jk} represents the one-dimensional Haar wavelet (38). Once again the dilation equations (37) govern the smoothing and differentiation transformations of the original configuration. – Referring to the previous paragraph, the difference information, i.e. the right column of Fig. 9, can be completely expanded in the two-dimensional Haar-wavelet basis (63). The correlations between the amplitudes of this multiresolution expansion are the wavelet correlations. Thus it becomes obvious that the wavelet correlations do characterize the subclustering of clumps.

This interpretation works not only for the Haar wavelet basis. In Fig. 10 a two-dimensional wavelet analysis has been performed with the so-called Daubechies D12 wavelet [11]. The decompositions (62) and (63) still hold; the only difference to the Haar-wavelet analysis is that 12 coefficients c_m are used in the dilation equations (37). Again the left column of Fig. 10 shows density plots of a sequence of D12-smoothed approximations to the original configuration, whereas the middle column represents density plots of the wavelet transform, which form a sequence of mutually orthogonal details. In order to provide a better picture of the subclustering aspect of the wavelet transform, the details at various scales are exhibited

again in the right column, but with only two gray-values: black for regions where the detail function becomes negative, indicating local voids, and white for regions where the detail function is positive, signaling the appearance of clusters at the various scales.

The borders between white and black regions in the right column of Fig. 10 are the zero-crossings of the wavelet transform. In signal analysis, these zero-crossings localize the signal sharp variation points at different scales and act as edge detectors [25]. The related wavelet transform maxima method [26] is used to compress and reconstruct signals efficiently.

4 Conclusions and outlook

For the complex dynamics of multiparticle processes we study the fundamental differential correlation densities in order to extract as much important statistical information (scaling, clustering) as possible. This has been demonstrated with the help of some simple discrete hierarchical branching models. An evolution equation for the (multivariate) generating function has been formulated, from which the correlation densities follow recursively. An integration over the correlation densities yields the moments.

Contrary to common practice, a careful conceptual distinction has to be made between forward (= theoretical) moments, which are obtained in evolution direction of the cascade process, and backward (= experimental) moments obtained in a backward analysis after the last cascade step has been performed. The former, often calculated from theory, can not be directly compared to the latter. Only the backward moments are accessible to experimental observation. We show that hierarchical branching processes with global density fluctuations on top of local density fluctuations do not show a rigorous scaling in the observed moments, although they are constructed by a purely selfsimilar iteration law. *In other words, a deviation from scaling of the observed moments does not necessarily imply that the cascade process is not selfsimilar.* This poses the provocative question: What is now the essence of a moment (multifractal) analysis which only focuses on scaling indices? A deviation from perfect scaling provides important information on the cascade mechanism as we have demonstrated by various models. – In contrast to a moment analysis, the selfsimilar (forward) scaling can still be extracted from the differential correlation densities.

However conventional correlation densities, especially higher orders, are hard to quantify. For their representation a compression is needed which reveals only the important (statistical) information and removes redundancy. In this respect the wavelet transformation appears very appealing. Wavelets constitute a selfsimilar and orthogonal basis; in addition, due to their multiresolution properties, they dissect structures into details (clumps) living on different scales. As a consequence the wavelet transformed correlation densities of all orders become very sparse for hierarchically organized processes. The second order correlations diagonalize completely in the Haar-wavelet basis. Furthermore, for any order of the wavelet correlations, the diagonal elements show scaling exponents which can be related to the multifractal dimensions. Beyond the multifractal analysis, those few off-diagonal elements of the higher order Haar-wavelet correlations which are nonzero and arise in bands show multiple

scaling and provide information on the subclustering structure of clumps such as correlations between small clusters living within larger clusters or voids.

This paper provides a demonstration of the wavelet capabilities for correlation studies. However, the techniques presented here have to be further refined in order to become a powerful tool for experimental analysis. In this respect wavelet packets [12, 13] might help to simplify the correlation structure even further. Also, in view of the recent success of correlation integrals [27], one should develop an analogue, so-called wavelet correlation integrals; this could set new standards of inferring information on clump correlations. Mathematically, there is the challenging problem to define the wavelet transformation on spheres which might have important applications in the context of angular intermittency.

We believe that a true understanding of the intermittency phenomenon in multiparticle cascading is still a long way off. For general particle cascades it is not a priori clear that they follow a hierarchical evolution structure. Particle cascades should be formulated as point processes, which need not be hierarchically organized from the beginning. The relevant splitting functions should dictate if hierarchical structures could evolve dynamically. This might be the case for perturbative QCD. Work in this direction, including the use of the wavelet transformation, is presently being carried out.

Apart from multiparticle physics, the above outlined wavelet technology might have applications in other fields of physics. For example, wavelet (clump) correlations may shed further light on the long-standing question in intermittent fully developed turbulence of how eddies decay at small scales and would allow to gain new and supplementary insight for phenomenological modeling. In disordered solid state systems multifractality and localization of electron wavefunctions are discussed; a wavelet analysis appears to be suitable [28]. Last not least, the simplifying aspect of wavelet correlations might ignite the century-old inverse problem of how to extract dynamics from correlations with new fuel.

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Figure Captions

Figure 1: One possible tree configuration with random branching. At each cascade step all existing branches, which have evolved so far, might independently split into a left and a right subbranch with probability $\tilde{p}$ or might not split with a probability $1-\tilde{p}$.

Figure 2: Energy curdling for a binary cascade with random branching shown for three cascade steps. After the last cascade step ($J=3$) each energy density is resolved at the finest scale.

Figure 3: Reduced two-bin correlation density $r_{k_1k_2} = \rho_{k_1k_2}/\rho_{k_1}\rho_{k_2}$ of the p-model (a), α-model (b), p-model with random branching (c) and the QCD-motivated cascade model (d). The splitting parameters and the number of cascade steps have been chosen to be $\alpha=\beta=\gamma_0=0.4$ and $J=6$, respectively; for case (c) the branching probability has been set to $\tilde{p}=0.5$.

Figure 4: Deviations from scaling of the backward moments $M_2(J,j)$ (upper two curves) and Haar wavelet correlations $\tilde{\rho}^{(J)}_{(j,k),(j,k)}$ (lower two curves) with respect to the scale index j for the p- (solid line) and α-model (dashed line). The number of cascade steps is $J=10$ and $\alpha=\beta=0.4$.

Figure 5: Deviations from scaling of the normalized forward moments $m_2(j)$ (dashed line), the normalized backward moments $M_2^{\mathrm{n}}(J,j) = M_2(J,j)/(\rho_0^{(J)})^2$ (solid line) and the normalized Haar wavelet correlations $\tilde{r}^{(J)}_{(jk),(jk)} = \tilde{\rho}^{(J)}_{(jk),(jk)}/(\tilde{\rho}_0^{(J)})^2$ (dotted line) with respect to the scale index j for the QCD-motivated cascade model. The number of cascade steps is $J=10$ and $\gamma_0=0.4$.

Figure 6: Reduced Haar wavelet transformed two-bin correlation density $\tilde{r}^{(J)}_{(j_1k_1),(j_2k_2)} = \tilde{\rho}^{(J)}_{(j_1k_1),(j_2k_2)}/(\tilde{\rho}_0^{(J)})^2$ of the p-model (a), α-model (b), p-model with random branching (c) and the QCD-motivated cascade model (d). The splitting parameters and the number of cascade steps have been chosen to be $\alpha=\beta=\gamma_0=0.4$ and $J=6$, respectively; for case (c) the branching probability has been set to $\tilde{p}=0.5$.

Figure 7: Deviations from scaling of $M_2(J,j)$ (upper curves) and $\tilde{\rho}^{(J)}_{(j,k)(j,k)}$ (lower curves) for the p-model with random branching with $J=10$, $\alpha=0.4$, $\tilde{p}=1$ (solid curve), $\tilde{p}=1/(1+\alpha^2)$ (dashed curve) and $\tilde{p}=0.5$ (dotted curve).

Figure 8: The p-model evolution of the energy densities at three adjacent cascade steps. The marked arrows represent the difference in energy density of neighbouring bins and are equal to the corresponding Haar wavelet amplitudes $2\tilde{\epsilon}_{jk}$. The clump correlations relate the fluctuations of e.g. arrow 1 with those of arrow 2.

Figure 9: Multiresolution analysis with the Haar wavelet basis of one particular two-dimensional α-model realization at scale $J = 6$ down to $j = 2$. Left column: sequence of smoothing operations. Middle column: difference between two adjacent smoothed scales. Right column: clump structure of middle column emphasized by reduction to two gray values (white/black) for positive/negative regions in the difference information.

Figure 10: Multiresolution analysis of 500 points from a Poisson transformed α-model realization in two dimensions with respect to the smooth compact Daubechies D12 wavelet. For further details see Fig. 9.

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